

STABILITY OF SECANT BUNDLES ON SECOND SYMMETRIC POWER OF CURVES

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ABSTRACT. Given a rank r stable bundle over a smooth irreducible projective curve C , there is an associated rank $2r$ bundle over $S^2(C)$, the second symmetric power of C . In this article we study the stability of this bundle. As a consequence we get an immersion from the moduli space of stable bundles over C to the associated moduli space of stable bundles over $S^2(C)$.

1. INTRODUCTION

Let C be a smooth, irreducible, projective curve of genus g over \mathbb{C} , the field of complex numbers, and let $S^n(C)$ denotes the n -fold symmetric power of C . Given a vector bundle E of rank r on C , one can associate a rank nr vector bundle $\mathcal{F}_n(E)$ on $S^n(C)$. This bundle was first studied by R. Schwarzenberger ([11]) for the case of line bundles, called it *secant bundle*. He used it to study the ring of rational equivalence class of $S^n(C)$. One natural question is to ask the (semi-) stability of $\mathcal{F}_n(E)$ with respect to some naturally chosen ample divisors. In [9], [5], [10], authors have studied this questions for the case of line bundles. In [7], authors have proved the semistability of $\mathcal{F}_2(E)$ (i.e. when $n = 2$) for some generic rank two stable bundles on C . On the other hand, in [6] and [4], authors have shown that the “morphism” from certain moduli space of (semi-)stable bundles over curves to an induced moduli space of (semi-)stable bundles over symmetric power of curves is (set-theoretically) injective (provided the later space is non-empty). In this article, we show that it is indeed non-empty when $n = 2$. More precisely, we have shown:

Theorem 1.1. *Let E be a rank r semi-stable bundle on C of degree $d \geq r$. Then the rank $2r$ bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is semi-stable with respect to the ample class $x + C$. And if $d > r$ and E is stable, then $\mathcal{F}_2(E)$ is stable with respect to $x + C$.*

As a consequence of the the above theorem, we get that the morphism from the moduli space $M_C(d, r)$ of semistable bundles over C to the moduli space $M_{S^2(C)}(c_1, c_2, H, 2r)$ of semistable bundles over $S^2(C)$, where the Chern classes c_1, c_2 depend on d and r , and $H = x + C$, is an immersion on the stable locus of the moduli space.

Remark 1.2. *In [2], it was shown that the bundle $\mathcal{F}_n(E)$ is parabolic semistable (resp. stable) whenever E is semistable (resp. E is non-trivial and stable), and together with [3], this shows that there is an injection from certain moduli space of stable bundles over curves to an induced moduli space of parabolic stable bundles over symmetric power of curves. The above result generalizes this.*

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2. PRELIMINARIES

Let C be a smooth, irreducible, projective curve of genus g over the field of complex numbers \mathbb{C} . On the cartesian product $C \times C$, we have a natural $\mathbb{Z}/2\mathbb{Z}$ action by means of the involution $\sigma : C \times C \rightarrow C \times C, (x, y) \mapsto (y, x)$. Let us denote the quotient space by $S^2(C)$. It is a smooth, irreducible, projective surface over \mathbb{C} , called the second symmetric power of curve. Let $\pi : C \times C \rightarrow S^2(C)$ be the quotient map. Recall that an effective divisor of degree two on C is of the form $x + y$, where $x, y \in C$. The set of all degree two effective divisors on C are naturally identified with $S^2(C)$.

Let q_1 and q_2 be the projections from $S^2(C) \times C$ to $S^2(C)$ and C respectively. Let

$$\Delta_2 := \{(D, z) \in S^2(C) \times C \mid z \in \text{Supp}(D)\} \subset S^2(C) \times C.$$

Then Δ_2 is a smooth divisor of $S^2(C) \times C$, called the universal effective divisor of degree 2 of C . Let $q : \Delta_2 \rightarrow S^2(C)$ be the restriction of q_1 . Then Δ_2 is a two-sheeted ramified cover of $S^2(C)$.

Let E be a rank r vector bundle on C . Define

$$\mathcal{F}_2(E) := q_*(q_2^*(E)|_{\Delta_2}).$$

Then $\mathcal{F}_2(E)$ is a rank $2r$ vector bundle on $S^2(C)$, called the secant bundle.

Remark 2.1. Suppose E is a line bundle on C of degree d . Then $\mathcal{F}_2(E)$ is a rank two vector bundle on $S^2(C)$. The Chern character of $\mathcal{F}_2(E)$ is given by

$$ch(\mathcal{F}_2(E)) = d(1 - e^{-x}) - (g - 1) + (g + 1 + \theta)e^{-x}$$

where x is the cohomology class of $x + C$ and θ is the cohomology class of the pull back of the theta divisor in $\text{Pic}^2(C)$ under the natural map $S^2(C) \rightarrow \text{Pic}^2(C)$ ([1, Chapter VIII, Lemma 2.5]). To find the Chern character of $\mathcal{F}_2(E)$ for any rank r vector bundle E , first one choose a filtration of E such that successive quotients are line bundles and then use the fact $\mathcal{F}_2(\oplus M_k) = \oplus \mathcal{F}_2(M_k)$ where M_k 's are line bundles over C . Then the Chern character of $\mathcal{F}_2(E)$ has the following form:

$$(1) \quad ch(\mathcal{F}_2(E)) = d(1 - e^{-x}) - r(g - 1) + r(g + 1 + \theta)e^{-x}$$

where $d = \text{degree}(E)$ ([2]).

Define a map $f : C \times C \rightarrow \Delta_2, (x, y) \mapsto (x + y, x)$. Then f is an identification. Let $p_i : C \times C \rightarrow C$ be the i -th coordinate projection, $i = 1, 2$. Then it is easy to see that $\mathcal{F}_2(E) = \pi_*(p_2^*(E))$. Now using the above remark (2.1) we see that the first Chern class of $\mathcal{F}_2(E)$ has the following expression:

$$c_1(\mathcal{F}_2(E)) = (d - r(g + 1))x + r\theta$$

where $d = \text{degree}(E)$, $r = \text{rank}(E)$. The cohomology group $H^4(S^2(C), \mathbb{Z})$ is naturally isomorphic to \mathbb{Z} and we have the following relations: $x^2 = 1, x.\theta = g, \theta^2 = g(g - 1)$.

Now we recall some definitions:

Definition 2.2. Let X be a smooth, irreducible, projective variety of dimension n over \mathbb{C} and let H be an ample divisor on X . Let E be a torsion free sheaf on X . The slope of E with respect to the ample divisor H , denoted by $\mu_H(E)$, is defined as $\mu_H(E) := \frac{\deg(E)}{\text{rank}(E)}$ where $\deg(E) := c_1(E).H^{n-1}$. A torsion free sheaf E on X is said to be semi-stable (with respect to the ample divisor H) if for every non-zero proper

subsheaf F of E we have $\mu_H(F) \leq \mu_H(E)$. E is said to be stable (with respect to the ample divisor H) if for every proper subsheaf F of E with $0 < \text{rank}(F) < \text{rank}(E)$, the above inequality is strict.

3. STABILITY OF SECANT BUNDLES

Let C be a smooth, irreducible, projective curve over \mathbb{C} of genus g and let E be a rank r vector bundle of degree d on C . In this section we will investigate the stability of the secant bundle $\mathcal{F}_2(E)$ on $S^2(C)$ with respect to the ample class $x + C$.

First we will recall some well-known results:

Lemma 3.1. *Let X, Y be two smooth, irreducible, projective varieties over \mathbb{C} and let $f : X \rightarrow Y$ be a finite morphism. Let \mathcal{E} be a vector bundle on Y and fix an ample divisor H on Y . Then \mathcal{E} is semi-stable with respect to the ample divisor H on Y if and only if $f^*(\mathcal{E})$ is semi-stable with respect to the ample divisor $f^*(H)$ on X .*

Proof. See [8, Chapter 3, Lemma 3.2.2]. \square

Lemma 3.2. *Let C be a smooth irreducible curve of genus $g \geq 1$ and let K_C be the canonical bundle of C . Let $J^{g-1}(C)$ be the variety of line bundles of degree $g-1$ of C , and let Θ be the divisor on $J^{g-1}(C)$ consisting of line bundles with non-zero sections. Let ξ be a line bundle on C of degree $g-3$ and*

$$\nu_\xi : C \times C \longrightarrow J^{g-1}(C)$$

be the morphism $(x, y) \mapsto \mathcal{O}_C(x + y) \otimes \xi$. Then

$$\nu_\xi^*(\Theta) \cong p_1^*(K_C \otimes \xi^*) \otimes p_2^*(K_C \otimes \xi^*) \otimes \mathcal{O}_{C \times C}(-\Delta)$$

where Δ is the diagonal of $C \times C$ and $p_i : C \times C \rightarrow C$ is the i -th coordinate projection.

Proof. See [9, Lemma 4.5]. \square

By Lemma (3.1), proving the semi-stability of $\mathcal{F}_2(E)$ on $S^2(C)$ with respect to $x + C$ is same as proving the semi-stability of $\pi^*\mathcal{F}_2(E)$ on $C \times C$ with respect to the ample divisor $H := \pi^*(x + C) = [x \times C + C \times x]$. The first Chern class of $\mathcal{F}_2(E)$ has the following expression: $c_1(\mathcal{F}_2(E)) = (d - r(g - 1))x + r\theta$. Now using Lemma (3.2), we see that

$$c_1(\pi^*\mathcal{F}_2(E)) = d[x \times C + C \times x] - r\Delta$$

where Δ is the diagonal of $C \times C$ and

$$\mu_H(\pi^*\mathcal{F}_2(E)) = \frac{d - r}{r}.$$

The vector bundles $\pi^*\mathcal{F}_2(E)$ and $p_1^*(E) \oplus p_2^*(E)$ are isomorphic outside the diagonal. On $C \times C$, these two bundles are related by the following exact sequence:

$$(2) \quad 0 \rightarrow \pi^*(\mathcal{F}_2(E)) \rightarrow p_1^*(E) \oplus p_2^*(E) \xrightarrow{q} E = p_1^*(E)|_\Delta = p_2^*(E)|_\Delta \rightarrow 0$$

where the homomorphism q is defined as $q : (u, v) \mapsto u|_\Delta - v|_\Delta$. From this exact sequence we get the following two exact sequences:

$$(3) \quad 0 \rightarrow p_1^*(E) \otimes \mathcal{O}_{C \times C}(-\Delta) \rightarrow \pi^*\mathcal{F}_2(E) \rightarrow p_2^*(E) \rightarrow 0,$$

and

$$(4) \quad 0 \rightarrow p_2^*(E) \otimes \mathcal{O}_{C \times C}(-\Delta) \rightarrow \pi^*\mathcal{F}_2(E) \rightarrow p_1^*(E) \rightarrow 0$$

[4, Section 3].

Theorem 3.3. *Let E be a rank r semi-stable bundle on C of degree $d \geq r$. Then the rank $2r$ bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is semi-stable with respect to the ample class $x + C$. And if $d > r$ and E is stable, then $\mathcal{F}_2(E)$ is stable with respect to $x + C$.*

Proof. Since the bundle $\pi^*\mathcal{F}_2(E)$ is pull back from $S^2(C)$, it is $\mathbb{Z}/2\mathbb{Z}$ equivariant. The $\mathbb{Z}/2\mathbb{Z}$ action on $C \times C$ lifts to an action on $p_1^*(E) \oplus p_2^*(E)$. The fibers of $p_1^*(E) \oplus p_2^*(E)$ are permuted in the same way as that of any element of $C \times C$. Also the injection $\pi^*\mathcal{F}_2(E) \hookrightarrow p_1^*(E) \oplus p_2^*(E)$ in the exact sequence (2) is $\mathbb{Z}/2\mathbb{Z}$ -equivariant. Note that the ample divisor $H = [x \times C + C \times x]$ is $\mathbb{Z}/2\mathbb{Z}$ -invariant. Let A be the maximal destabilizing subsheaf of $\pi^*\mathcal{F}_2(E)$ of rank s . Then A is reflexive and hence locally free. Then A has a $\mathbb{Z}/2\mathbb{Z}$ -linearization such that the inclusion $A \hookrightarrow \pi^*\mathcal{F}_2(E)$ is $\mathbb{Z}/2\mathbb{Z}$ -equivariant (for a proof see [10, Proposition 4.2.2]). Thus the composition of inclusions

$$(5) \quad \psi : A \hookrightarrow \pi^*\mathcal{F}_2(E) \hookrightarrow p_1^*E \oplus p_2^*E$$

is also $\mathbb{Z}/2\mathbb{Z}$ -equivariant. Let $\text{pr}_i : p_1^*E \oplus p_2^*E \rightarrow p_i^*E$ be the i -th coordinate projection and $\psi_i = \text{pr}_i \circ \psi, i = 1, 2$. Let $s_1 = \text{rank}(\text{Ker}\psi_1)$ and $s_2 = \text{rank}(\text{Im}\psi_1)$. Due to $\mathbb{Z}/2\mathbb{Z}$ -equivariance, we have $s_1 = \text{rank}(\text{Ker}\psi_2)$ and $s_2 = \text{rank}(\text{Im}\psi_2)$. Also notice that $\text{Ker}\psi_1 \subseteq \text{Im}\psi_2$ and $\text{Ker}\psi_2 \subseteq \text{Im}\psi_1$. This shows that $s_1 \leq s_2$.

First assume $s_1 = 0$. Then both ψ_1 and ψ_2 are inclusions. Consider the following inclusions

$$\wedge^s \psi_1 : \wedge^s A \hookrightarrow \wedge^s p_1^*E$$

and

$$\wedge^s \psi_2 : \wedge^s A \hookrightarrow \wedge^s p_2^*E.$$

Restricting the first inclusion to the curves of the form $x \times C$ and the second inclusion to the curves of the form $C \times x$ we get that $\deg(A) = \deg(\wedge^s A) \leq 0$. This gives a contradiction that A is a maximal destabilizing subsheaf as $\mu_H(A) > \mu_H(\pi^*\mathcal{F}_2(E)) \geq 0$.

Now assume $s_1 > 0$. Restricting the exact sequence (3) to the curves of the form $C \times x$ we get an exact sequence

$$0 \rightarrow E \otimes \mathcal{O}_C(-x) \rightarrow \pi^*\mathcal{F}_2(E)|_{C \times x} \rightarrow \mathcal{O}_C^{\oplus r} \rightarrow 0.$$

Let A_2 (respectively, A_1) be the image (respectively, kernel) of the induced map $A|_{C \times x} \rightarrow \mathcal{O}_C^{\oplus r}$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E \otimes \mathcal{O}_C(-x) & \longrightarrow & \pi^*\mathcal{F}_2(E)|_{C \times x} & \longrightarrow & \mathcal{O}_C^{\oplus r} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A_1 & \longrightarrow & A|_{C \times x} & \longrightarrow & A_2 \longrightarrow 0 \end{array}$$

where the rows are exact and the vertical arrows are injections. Note that, $\text{rank}(A_1) = s_1$ and $\text{rank}(A_2) = s_2$. Since E is semistable (resp. stable), $\mu(A_1) \leq \frac{d-r}{r}$ (resp. $(<)$) and $\mu(A_2) \leq 0$. Combining these two we get that $\deg(A|_{C \times x}) \leq \frac{s_1(d-r)}{r}$ (resp. $(<)$). Similarly, using the exact sequence (4), we get that $\deg(A|_{x \times C}) \leq \frac{s_1(d-r)}{r}$ (resp. $(<)$). Thus $\deg(A) \leq \frac{2s_1(d-r)}{r}$ (resp. $(<)$). From the discussion above, we

see that, $2s_1 \leq s_1 + s_2 = s$. Hence $\mu_H(A) \leq \frac{d-r}{r}$ (resp. $(<)$), a contradiction. This shows that $\pi^*\mathcal{F}_2(E)$ is semi-stable with respect to the ample class H .

Now let \tilde{A} be a proper subsheaf of $\mathcal{F}_2(E)$, where E is stable and $d > r$. Taking double dual, if necessary, we can assume that \tilde{A} is a vector bundle. Then $A := \pi^*(\tilde{A})$ is a proper subsheaf of $\pi^*\mathcal{F}_2(E)$ and the inclusion $A \hookrightarrow \pi^*\mathcal{F}_2(E)$ is $\mathbb{Z}/2\mathbb{Z}$ -equivariant. Now we proceed as above to conclude that $\mu_H(A) < \mu_H(\pi^*\mathcal{F}_2(E))$. Hence $\mathcal{F}_2(E)$ is stable with respect to the ample class $x + C$. \square

Remark 3.4. Let $\mathcal{M}_1 := M_C(r, d)^s$ denotes the moduli space of stable bundles on C of rank r and degree d , and let $\mathcal{M}_2 := M_{S^2(C)}(c_1, c_2, H, 2r)^s$ denotes the moduli space of stable bundles over $S^2(C)$, where the Chern classes c_1, c_2 depend d and r and can be computed via equation(1), and $H = x + C$. Assume $d > r$. Then by the above theorem we get that the moduli space \mathcal{M}_2 is non-empty and we have a morphism

$$\Phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2, \quad E \mapsto \mathcal{F}_2(E).$$

Since $d > r$, we have $\mu(E) > 1$ for every E in \mathcal{M}_1 . Now using [6, Theorem 2.1], we see that the morphism Φ is injective. Let us denote by $T_{[E]}$, the Zariski tangent space of \mathcal{M}_1 at $[E]$, and by $T_{[\mathcal{F}_2(E)]}$, the Zariski tangent space of \mathcal{M}_2 at $[\mathcal{F}_2(E)]$. Then $T_{[E]} \cong \text{Ext}^1(E, E)$ and $T_{[\mathcal{F}_2(E)]} \cong \text{Ext}^1(\mathcal{F}_2(E), \mathcal{F}_2(E))$. The induced map

$$d\Phi : T_{[E]} \rightarrow T_{[\mathcal{F}_2(E)]}$$

is given by

$$(0 \rightarrow E \rightarrow E' \rightarrow E \rightarrow 0) \mapsto (0 \rightarrow \mathcal{F}_2(E) \rightarrow \mathcal{F}_2(E') \rightarrow \mathcal{F}_2(E) \rightarrow 0).$$

Since E' is semi-stable and $\mu(E') > 1$, again using [6, Theorem 2.1] we see that the map $d\Phi$ is injective. From this we conclude that the morphism Φ is an immersion.

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